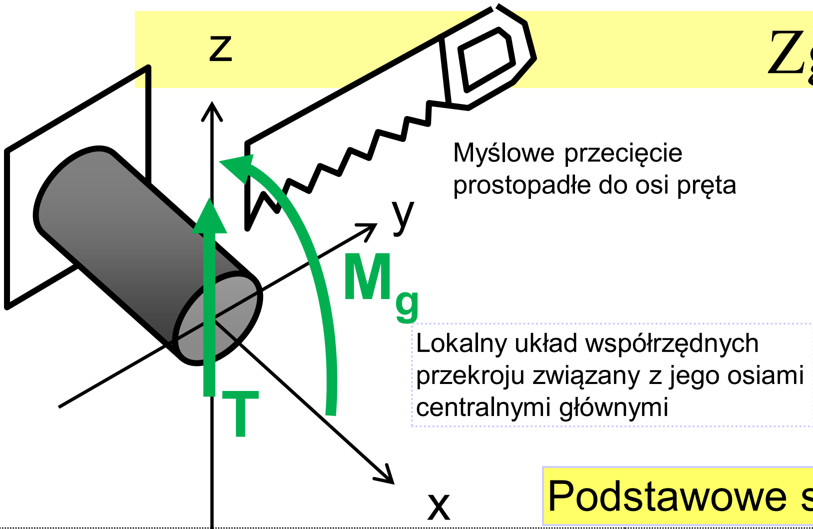
Wydział Mechaniczny Energetyki i Lotnictwa
Zakład Wytrzymałości Materiałów i Konstrukcji

Wykład 9

Pręty zginane – belki

Wyznaczanie rozkładu sił wewnętrznych

Zginanie proste



Występują składowe wysiłku przekroju:

T – Siła tnąca

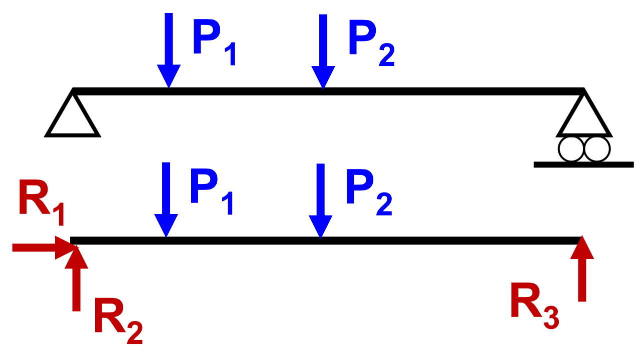
$\mathbf{M_g}$ – Moment gnący

Jeśli **T** = 0 to mamy zginanie czyste

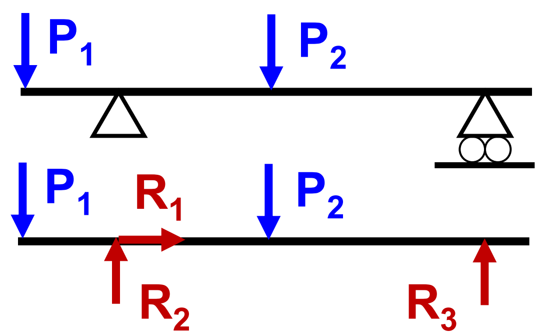
Jeśli **T** ≠ 0 to mamy zginanie poprzeczne

Podstawowe schematy belek statycznie wyznaczalnych:

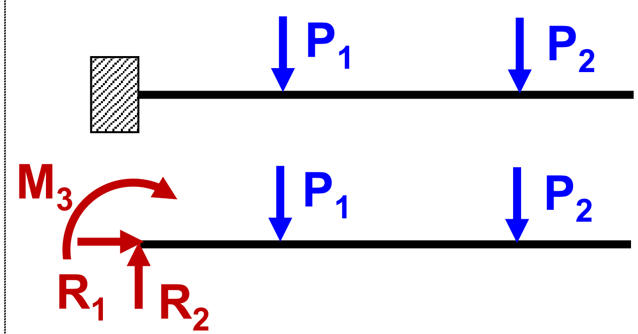
a) dwupodporowa



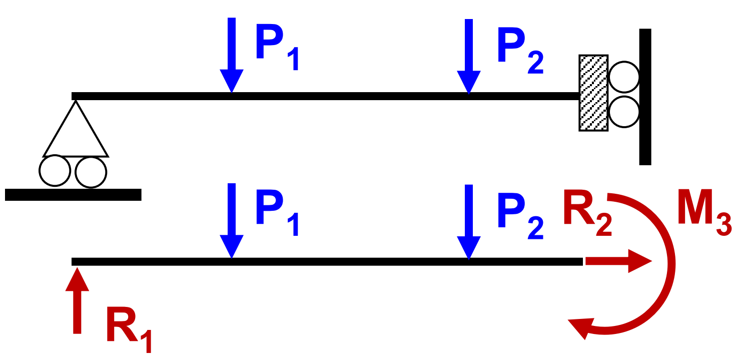
b) dwupodporowa z wysięgnikiem



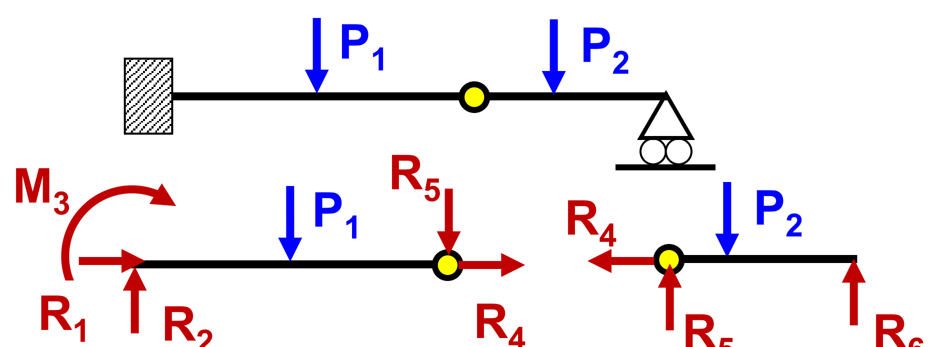
c) wspornikowa



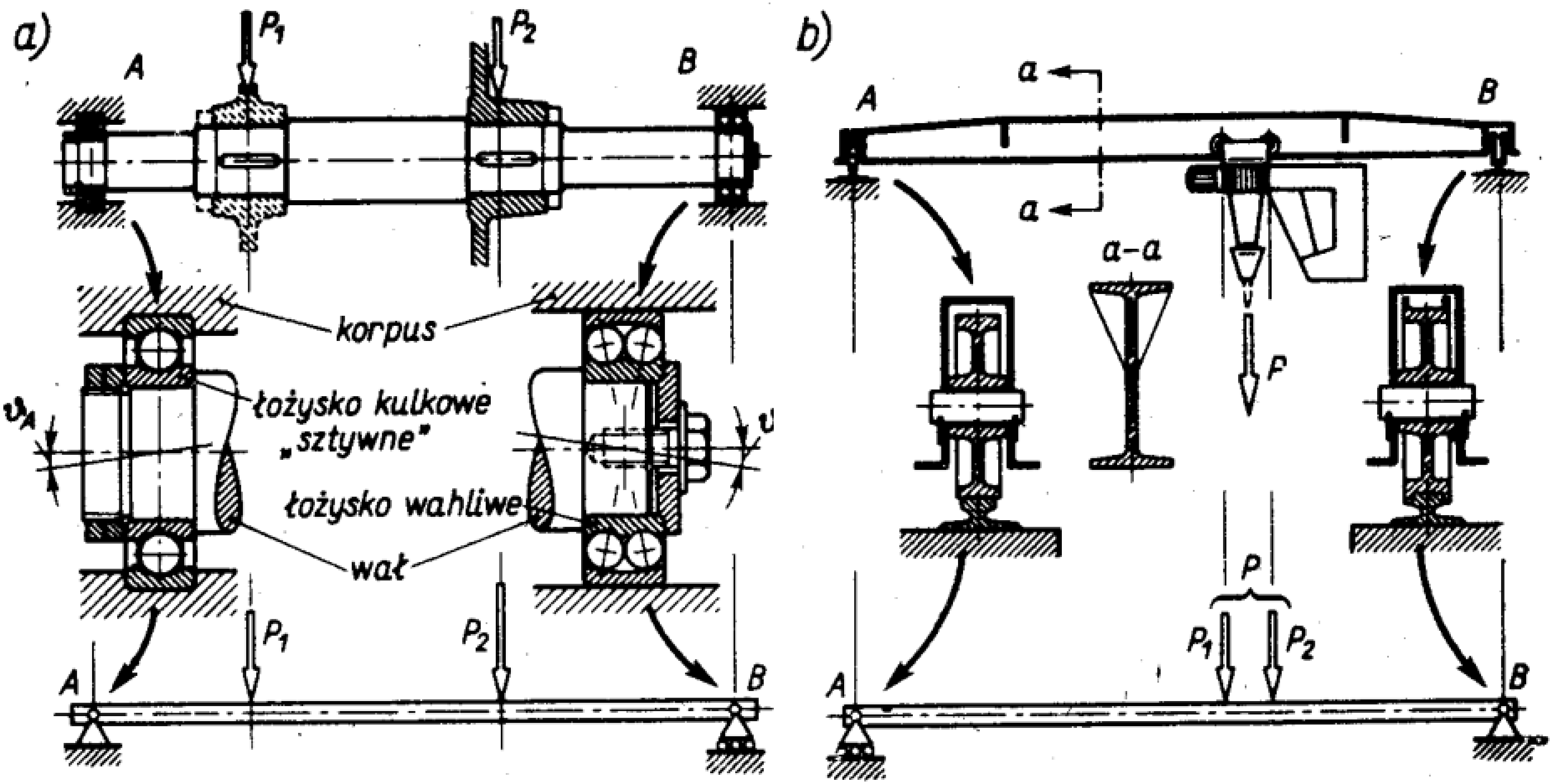
d) z wózkiem odbierającym moment



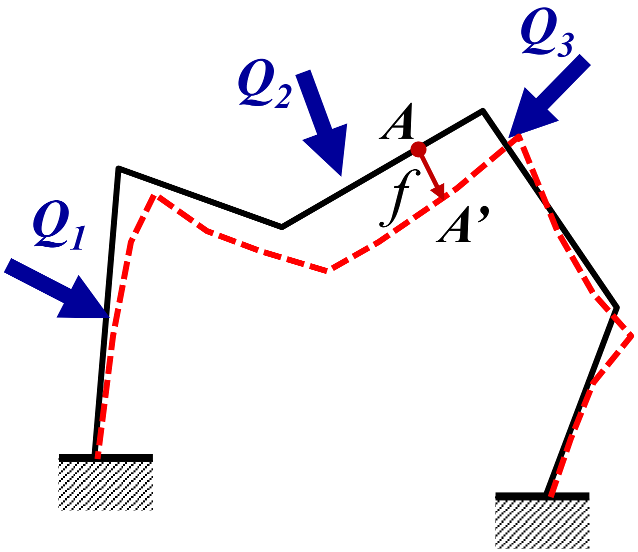
e) z przegubem



Przykłady tworzenia schematu obliczeniowego



Zasada superpozycji



Konstrukcja liniowa : Przemieszczenia i naprężenia są liniowymi funkcjami obciążeń

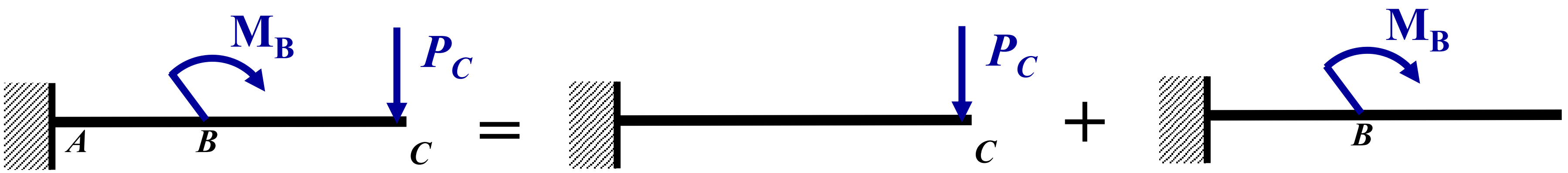
$$f = \sum_{j=1}^{n} \alpha_j Q_j \qquad \sigma = \sum_{j=1}^{n} \beta_j Q_j$$

α_j , β_j - współczynniki liniowe

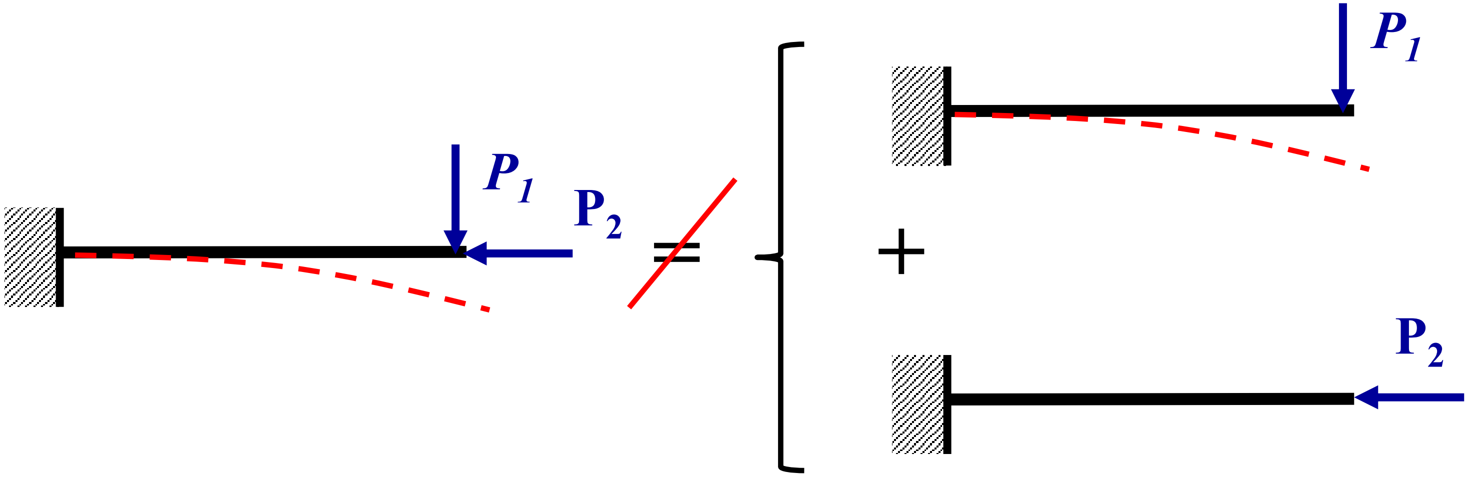
Warunki: 1) Materiał liniowo-sprężysty
2) Odkształcenia są małe
3) Przemieszczenia są małe

Zasada zesztywnienia (wymiary początkowe)

Zasada superpozycji *(niezależności działania obciążeń)*
W przypadku, gdy na konstrukcję liniową działa złożony układ obciążeń, to skutek działania tego układu jest równy sumie skutków obciążeń składowych

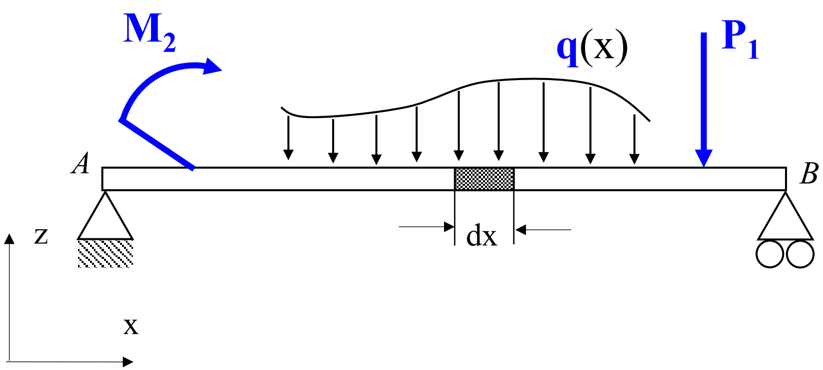


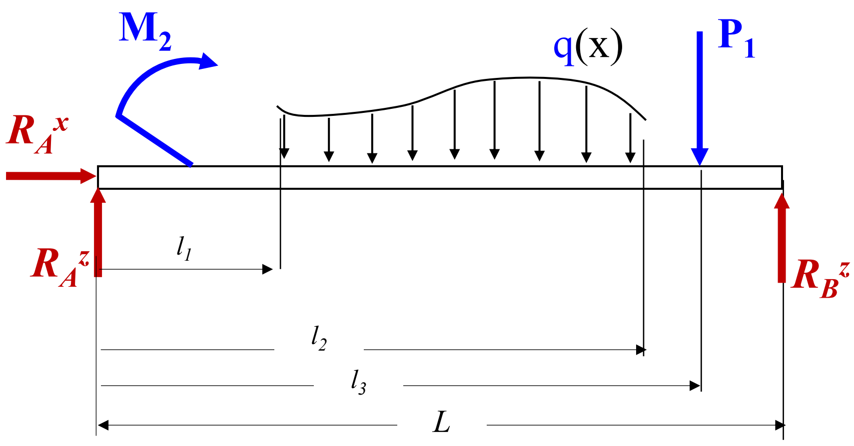
Nieraz nie można zastosować zasady superpozycji



Wyznaczanie składowych wysiłku przekroju w prostym zginaniu

Zginanie to taki przypadek obciążenia pręta, w którym występuje siła tnąca T i moment gnący Mg





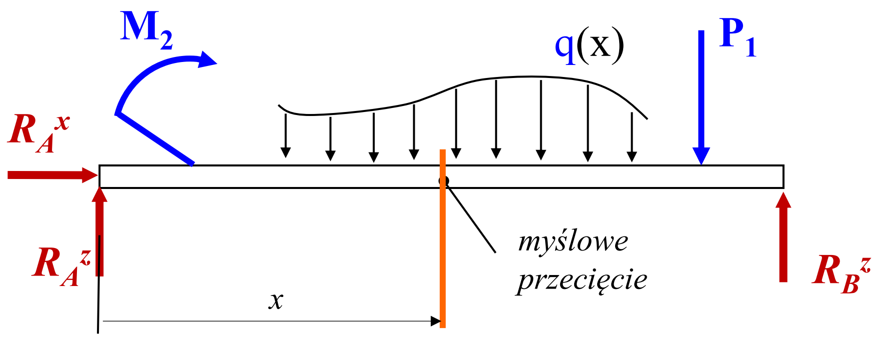
Zwykle zadanie zaczynamy od uwolnienia belki od więzów i wyznaczenia reakcji:

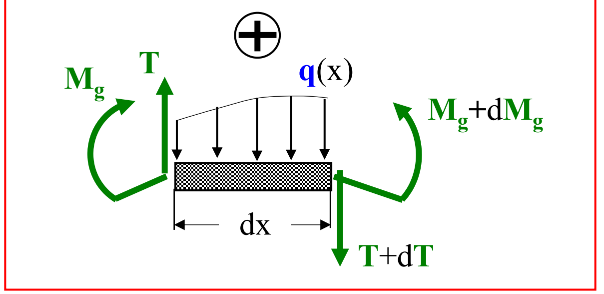
$$\sum F_x = 0: \quad R_A^x = 0$$

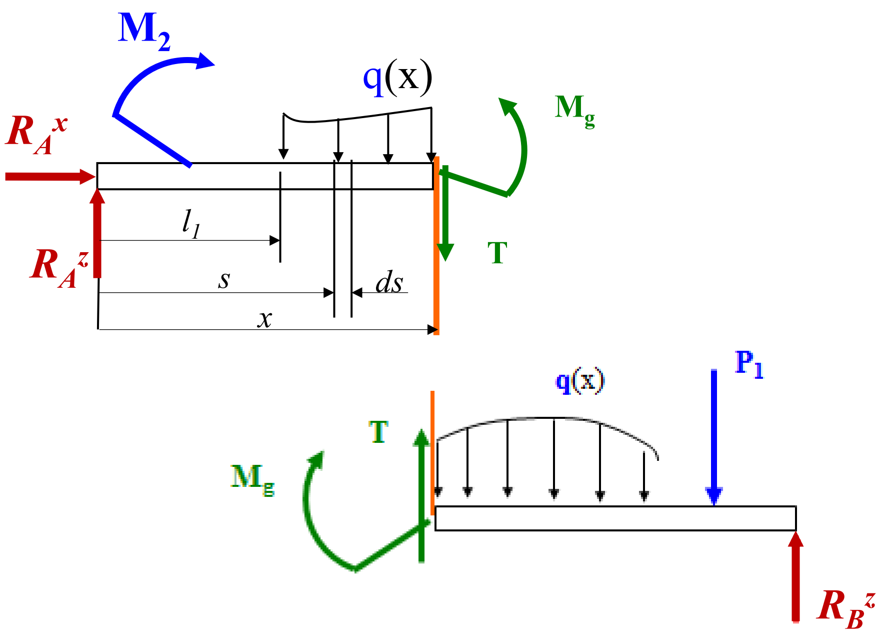
$$\sum M_A = 0: \quad R_B^z \cdot L - P_1 \cdot l_3 - \int_{l_1}^{l_2} q(x) \cdot x\, dx - M_2 = 0$$

$$\sum F_z = 0: \quad R_A^z + R_B^z - P_1 - \int_{l_1}^{l_2} q(x)\, dx = 0$$

Wyznaczanie składowych wysiłku przekroju w prostym zginaniu







Poszukiwane funkcje Mg(x) i T(x) możemy wyznaczyć z warunku równowagi myślowo wyciętej części (np. lewej):

$$\sum F_z = 0: \quad R_A^z - T(x) - \int_{l_1}^{x} q(s)\, ds = 0$$

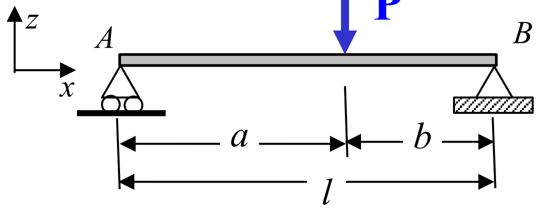
$$\sum M_{pp} = 0:$$

$$-R_A^z \cdot x + \int_{l_1}^{x} q(s) \cdot (x - s)\, ds - M_2 + M_g(x) = 0$$

Warto pamiętać, że pomiędzy funkcjami Mg(x), T(x) i q(x) zachodzą związki:

$$T(x) = \frac{dM_g(x)}{dx} \qquad q(x) = -\frac{dT(x)}{dx}$$

Zadanie 1



Równania równowagi:

$$\sum F_x = 0 \rightarrow R_B^x = 0$$

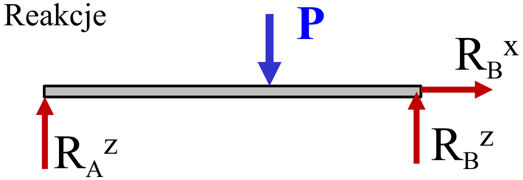
$$\sum M_A = 0 \rightarrow R_B^z \cdot l - P \cdot a = 0$$

$$R_B^z = P \cdot a/l$$

$$\sum M_B = 0 \rightarrow -R_A^z \cdot l + P \cdot b = 0$$

$$R_A^z = P \cdot b/l$$

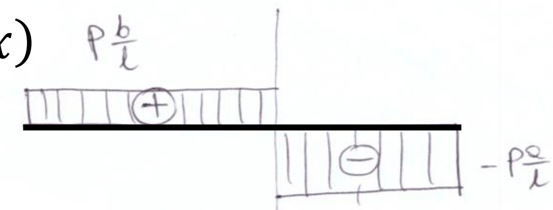
Reakcje



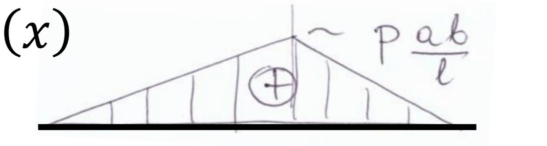
Wyniki:

$$T(x) = \frac{dM_g(x)}{dx}$$

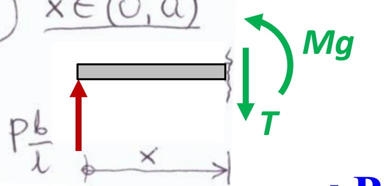
$T(x)$



$M_g(x)$



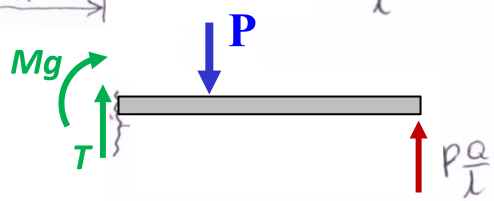
(I) $x \in (0, a)$



Równania równowagi:

$$\sum F_z = 0:$$

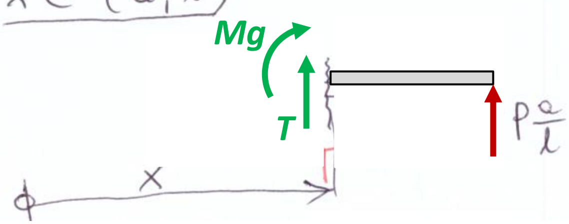
$$P\frac{b}{l} - T = 0 \rightarrow T = P\frac{b}{l}$$



$$\sum M_{pp} = 0: \quad -P\frac{b}{l} \cdot x + M_g = 0$$

$$M_g = P\frac{b}{l} \cdot x$$

(II) $x \in (a, l)$

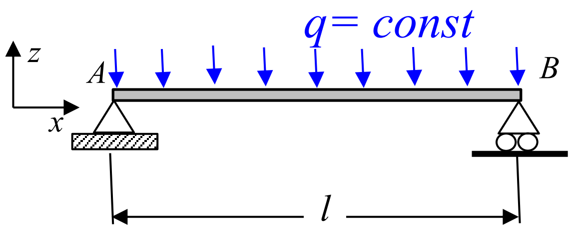


$$\sum F_z = 0: \quad T + P\frac{a}{l} = 0 \Rightarrow T = -P\frac{a}{l}$$

$$\sum M_{pp} = 0: \quad P\frac{a}{l} \cdot (l - x) - M_g = 0$$

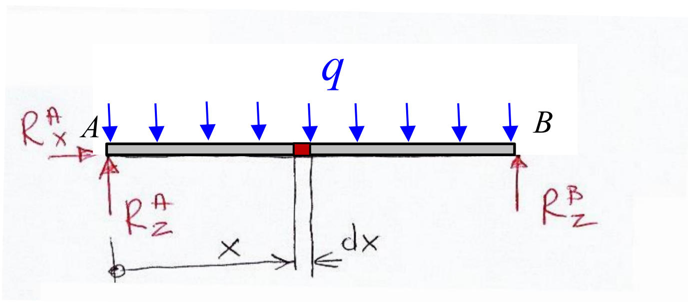
$$M_g = P\frac{a}{l}(l - x)$$

Zadanie 2



Reakcje:

$$\sum M_A = 0 : \qquad R_B^z\, l - \int_0^l x\, q\, dx = 0$$

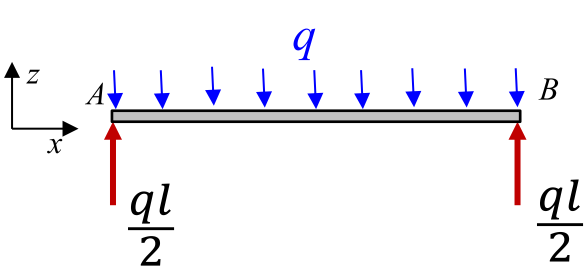


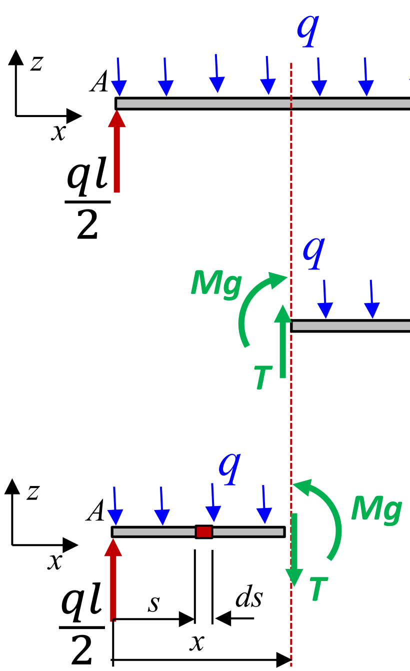
$$R_B^z = \frac{1}{l}\, q\, \frac{1}{2} x^2 \Big|_0^l \quad \rightarrow \quad \boxed{R_B^z = \frac{ql}{2}}$$

$$\underline{\sum F_z = 0 :} \qquad R_z^A + R_z^B - \int_0^l q\, dx = 0$$

$$R_z^A = q x \Big|_0^l - \frac{ql}{2}$$

$$\boxed{R_z^A = \frac{ql}{2}}$$





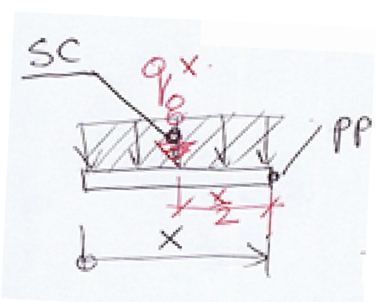
Równania równowagi:

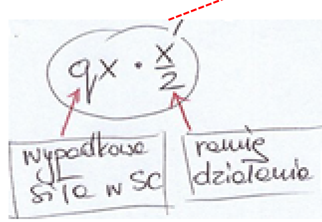
$$\sum F_z = 0 : \qquad \tfrac{ql}{2} - \int_0^x q\, ds \; - T = 0 \;\rightarrow\; \boxed{T = q\left(\tfrac{l}{2} - x\right)}$$

$$\sum M_{pp} = 0 : \qquad -\tfrac{ql}{2} \cdot x + \int_0^x (x-s)\, q ds + M_g = 0$$

$$\rightarrow\; M_g = \tfrac{ql}{2} \cdot x - q \int_0^x (x-s)\, ds \;\rightarrow\; M_g = \tfrac{ql}{2}\, x - q\left(xs - \tfrac{1}{2}s^2\right)\Big|_0^x$$

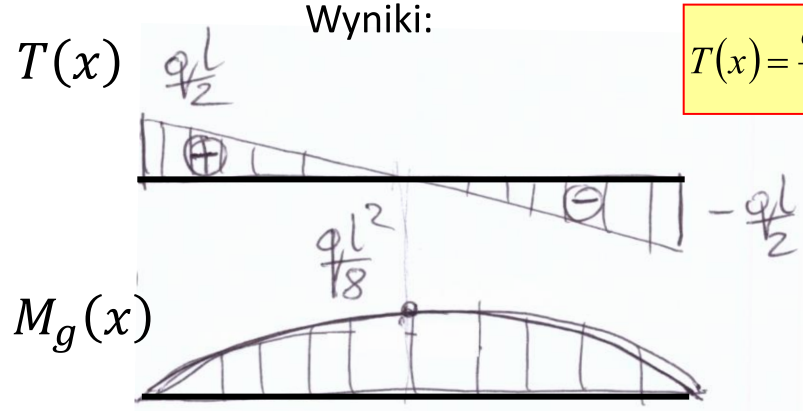
$$\rightarrow\; M_g = \tfrac{ql}{2}\, x - q\left(x^2 - \tfrac{1}{2}x^2\right) \;\rightarrow\; M_g = \tfrac{ql}{2}\, x - \tfrac{1}{2}qx^2 \;\rightarrow\; \boxed{M_g = \frac{1}{2}\, qx(l-x)}$$

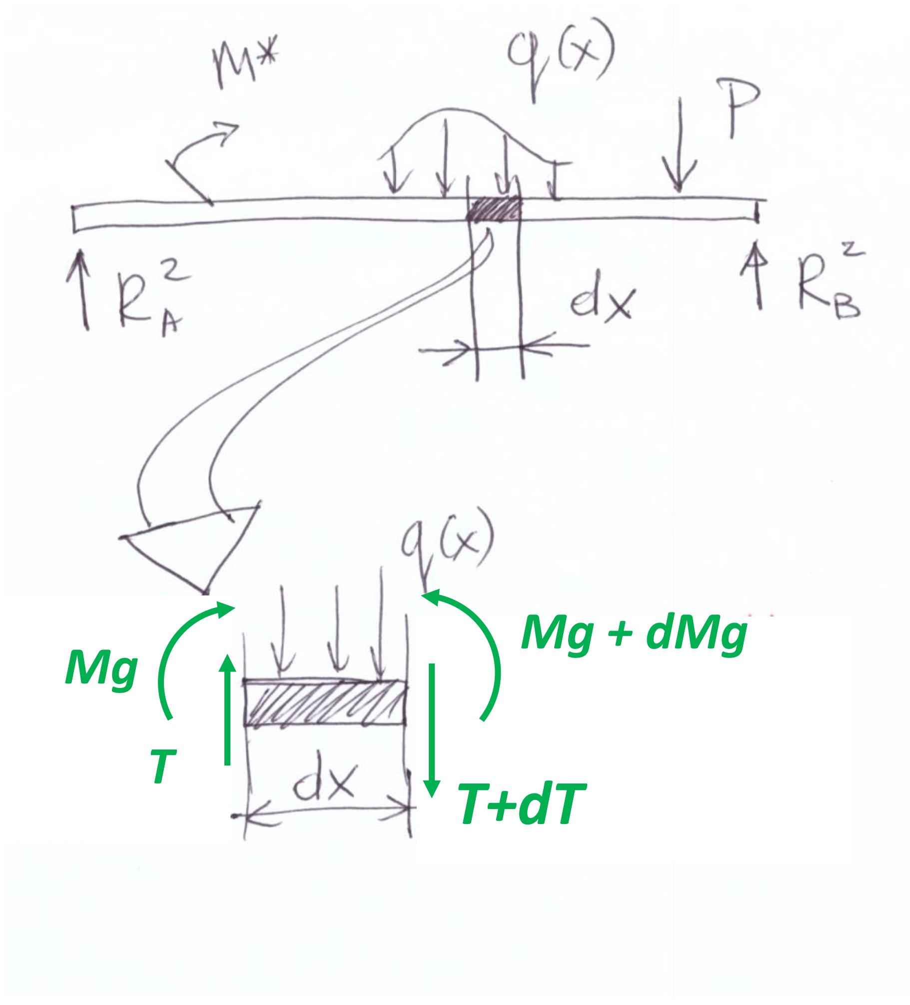




Wyniki:

$$T(x) = \frac{dM_g(x)}{dx}$$

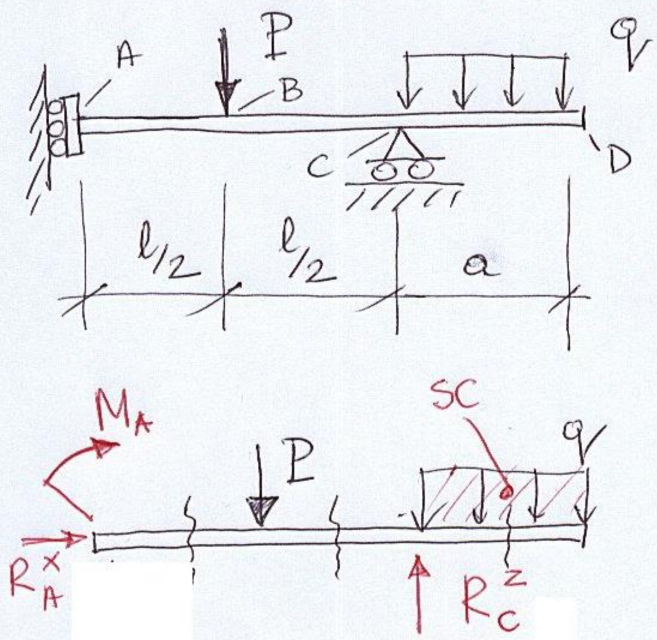




$$q(x) = -\frac{dT(x)}{dx}$$

$$T(x) = \frac{dM_g(x)}{dx}$$

Zadanie 3



$P = 2\,kN$

$q = 10\,\frac{kN}{m}$

$l = 1\,m$

$a = 0.4\,m$

$T(x),\ M_g(x)\ ?$

Reakcje:

r-nie r-gi:

$\Sigma F_x = 0:$ $\quad R_A^x = 0$

$\Sigma M_c = 0: \quad -M_A + P \cdot \frac{l}{2} - q a \cdot \frac{a}{2} = 0$

$$M_A = \frac{Pl}{2} - \frac{qa^2}{2}$$

$$M_A = \frac{2 \cdot 1}{2} - \frac{10 \cdot 0.4^2}{2}$$

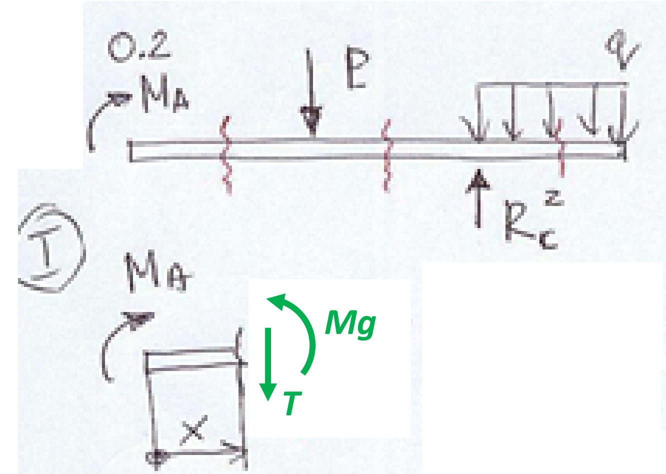
$$M_A = 1 - 0.8 = 0.2\ kNm$$

$\Sigma F_z = 0$

$$-P + R_c^z - qa = 0$$

$$R_c^z = qa + P$$

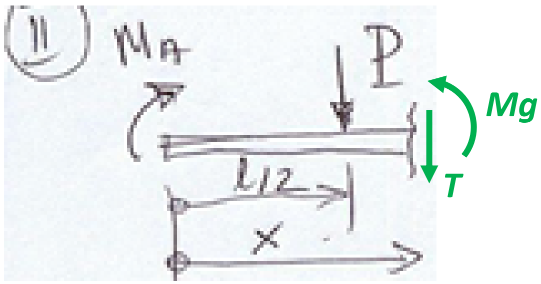
$$R_c^z = 10 \cdot 0.4 + 2 = 6\ kN$$



Myślowe przecięcia:

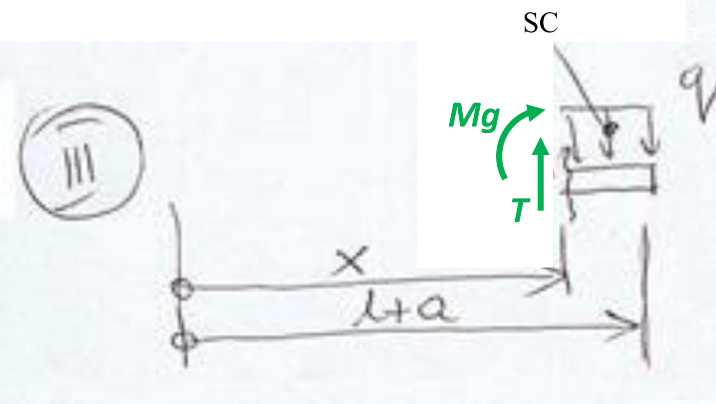
$$\Sigma F_z = 0: \quad -T = 0 \qquad \boxed{T = 0}$$

$$\Sigma M_{pp} = 0: \quad -M_A + M_g = 0 \qquad \boxed{M_g(x) = M_A}$$



$$\Sigma F_z = 0: \quad -P - T = 0 \qquad \boxed{T(x) = -P}$$

$$\Sigma M_{pp} = 0: \quad -M_A + P\left(x - \frac{l}{2}\right) + M_g = 0 \qquad \boxed{M_g(x) = M_A + P\left(\frac{l}{2} - x\right)}$$

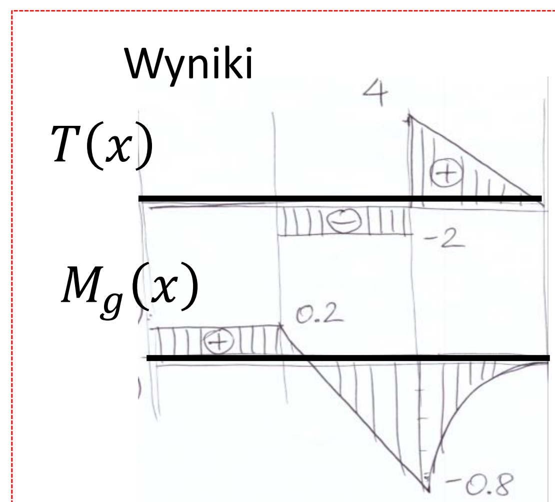


$$\Sigma F_z = 0: \quad T - q\cdot(l+a-x) = 0 \qquad \boxed{T(x) = q\,(l+a-x)}$$

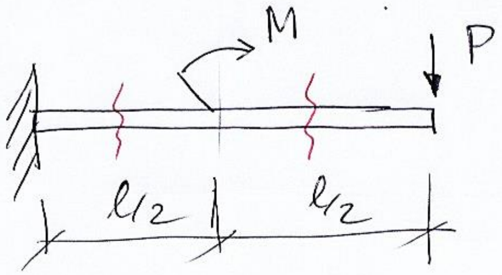
$$\Sigma M_{pp} = 0:$$

$$-M_g - q\cdot\frac{(l+a-x)^2}{2} = 0$$

$$\boxed{M_g(x) = -q\,\frac{(l+a-x)^2}{2}}$$



Zadanie 4



$M = 0.5$ kNm

$P = 1$ kN

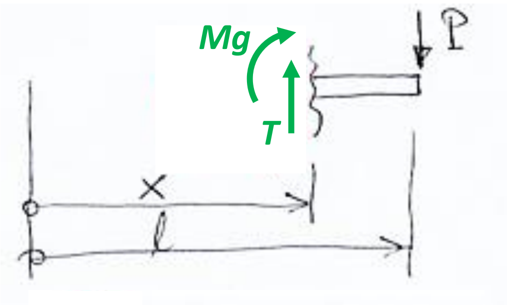
$l = 1$ m

$M_g(x)$

$T(x)$

2

<u>r-nie r-gi:</u>

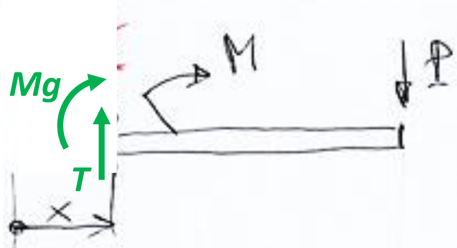


$\Sigma F_z = 0: \quad T - P = 0$

$$T(x) = P$$

$\Sigma M_{PP} = 0: \quad -M_g - P \cdot (l - x) = 0$

$$M_g(x) = -P(l - x)$$

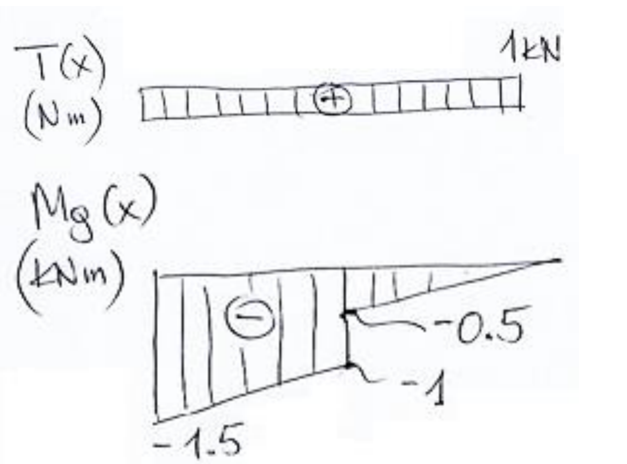


$\Sigma F_z = 0: \quad T - P = 0$

$$T(x) = P$$

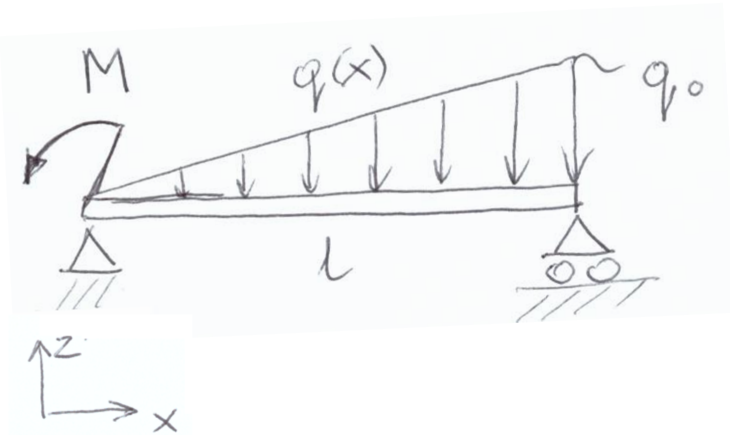
$\Sigma M_{PP} = 0: \quad -M_g - M - P(l - x) = 0$

$$M_g(x) = -P(l - x) - M$$



Wyniki

Zadanie 5

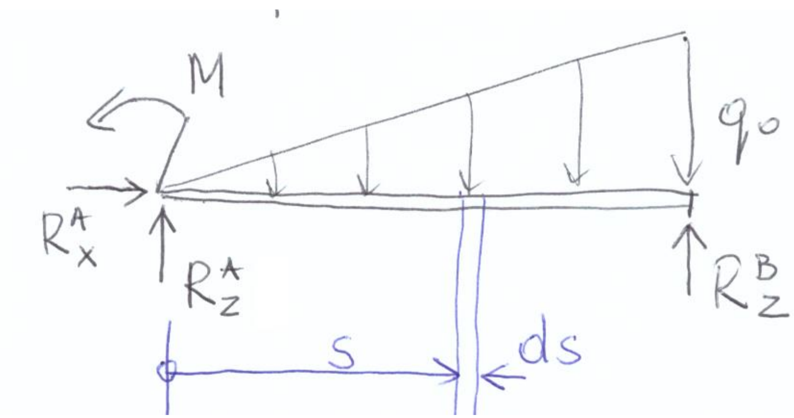


$$q_0 = 9 \ \frac{kN}{m}$$

$$M = 1{,}25 \ kNm$$

$$l = 1 \ m$$

$$q(x) = \frac{q_0}{l} \cdot x$$



Reakcje: $\Sigma M_A = 0$:

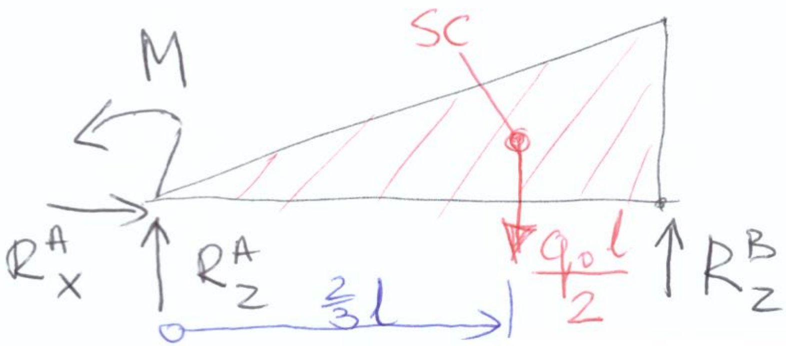
$$-\int_0^l s \cdot q(s) \cdot ds + R_z^B \cdot l + M = 0$$

$$R_z^B = \int_0^l \frac{q_0}{l^2} \cdot s^2 ds - \frac{M}{l}$$

$$R_z^B = \frac{q_0}{l^2} \frac{1}{3} l^3 - \frac{M}{l}$$

$$R_z^B = \frac{q_0 l}{3} - \frac{M}{l}$$

Prościej (graficznie)



$$\Sigma M_A = 0:$$

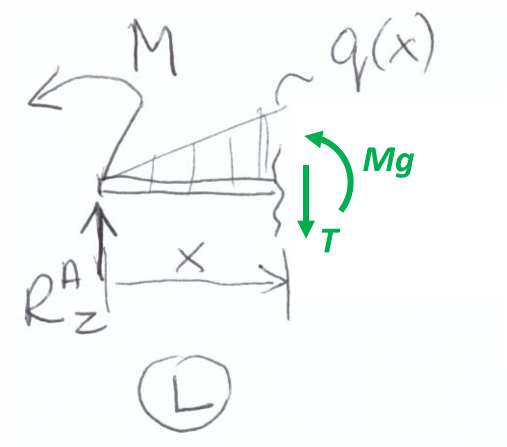
$$-\frac{q_0 l}{2} \cdot \frac{2}{3} l + R_z^B \cdot l + M = 0$$

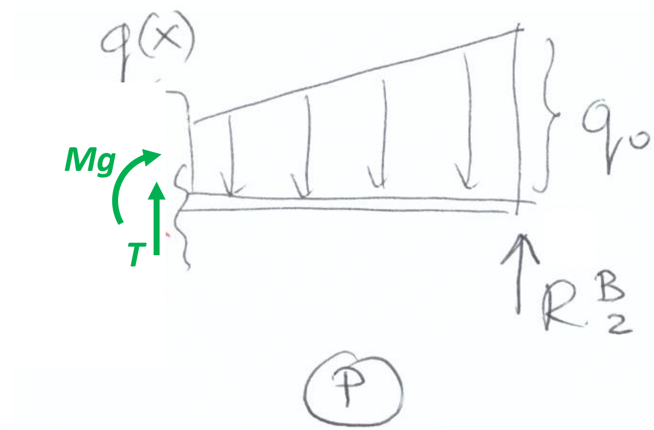
$$R_z^B = \frac{q_0 l}{3} - \frac{M}{l} = 1{,}75 \text{ kN}$$

$$\Sigma M_B = 0: \quad \frac{q_0 l}{2} \cdot \frac{1}{3} l - R_z^A \cdot l + M = 0$$

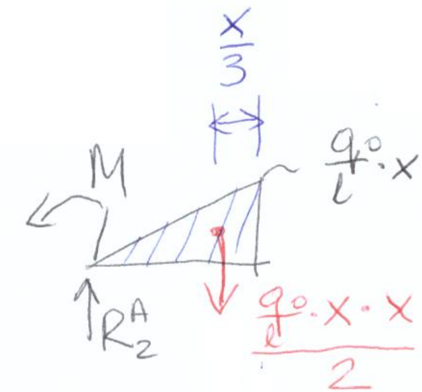
$$R_z^A = \frac{q_0 l}{6} + \frac{M}{l} = 2{,}75 \text{ kN}$$

$$\Sigma F_x = 0: \quad R_x^A = 0$$





$$\Sigma F_z = 0: \quad -T + R_z^A - \int_0^x q(s)\,ds = 0$$



$$T = \left(\frac{q_0 l}{6} + \frac{M}{l}\right) - \frac{\frac{q_0}{l}\cdot x \cdot x}{2} = \frac{q_0 l}{6} + \frac{M}{l} - \frac{q_0 x^2}{2l}$$

graficznie

$$\Sigma M_{pp} = 0: \quad M_g + \int_0^x (x-s)\,q(s)\cdot ds - R_z^A \cdot x + M = 0$$

graficznie

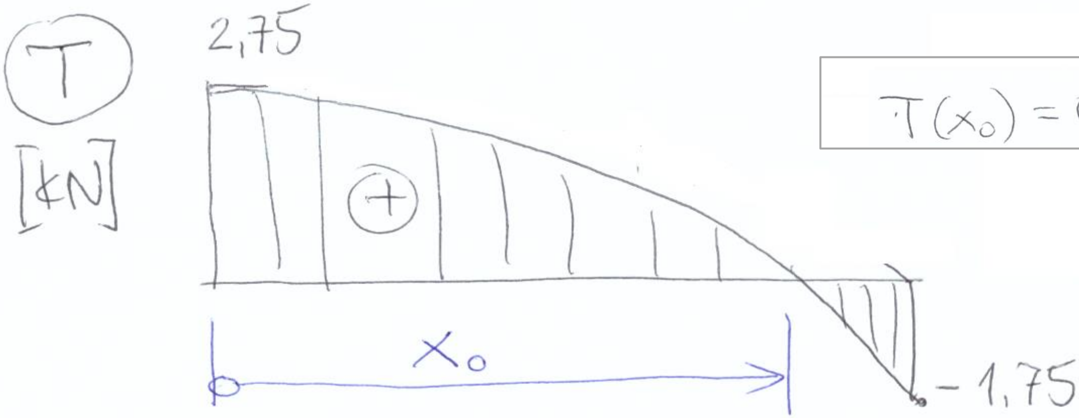
$$M_g = \left(\frac{q_0 l}{6} + \frac{M}{l}\right)\cdot x - M - \frac{\frac{q_0}{l}\, x \cdot x}{2}\cdot\frac{x}{3}$$

$$M_g = -\frac{q_0 x^3}{6l} + \left(\frac{q_0 l}{6} + \frac{M}{l}\right)\cdot x - M$$

$$T = \frac{q_0 l}{6} + \frac{M}{l} - \frac{q_0 x^2}{2l}$$

$$T(0) = \frac{q_0 l}{6} + \frac{M}{l} = \frac{9 \cdot 1}{6} + \frac{1{,}25}{1} = 2{,}75 \text{ kN} = R_z^A$$

$$T(l) = \frac{q_0 l}{6} + \frac{M}{l} - \frac{q_0 l}{2} = -1{,}75 \text{ kN} = -R_z^B$$



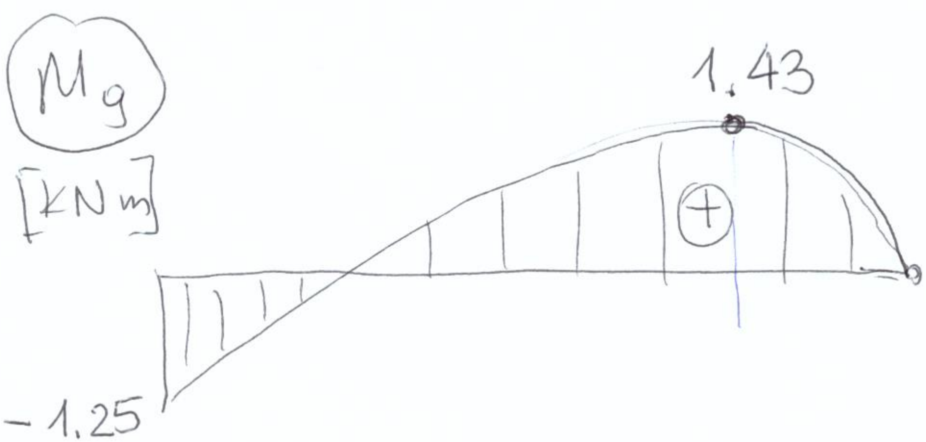
$$T(x_0) = 0 \rightarrow \frac{q_0 l}{6} + \frac{M}{l} - \frac{q_0 x_0^2}{2l} = 0$$

$$x_0 = \pm\sqrt{\left(\frac{q_0 l}{6} + \frac{M}{l}\right) \cdot \frac{2l}{q}}$$

$$x_0 = \pm\sqrt{\left(\frac{9 \cdot 1}{6} + \frac{1{,}25}{1}\right) \cdot \frac{2 \cdot 1}{9}}$$

$$x_0 = 0.782 \text{ m}$$

$$M_g = -\frac{q_0 x^3}{6l} + \left(\frac{q_0 l}{6} + \frac{M}{l}\right) \cdot x - M$$

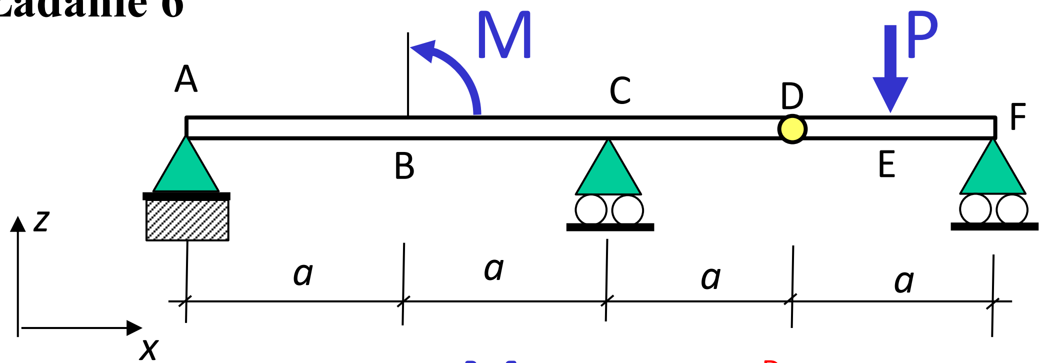


$$M_g(0) = -M = -1{,}25 \text{ kNm}$$

$$M_g(l) = 0$$

$$M_g(x_0) = 1{,}43 \text{ kNm}$$

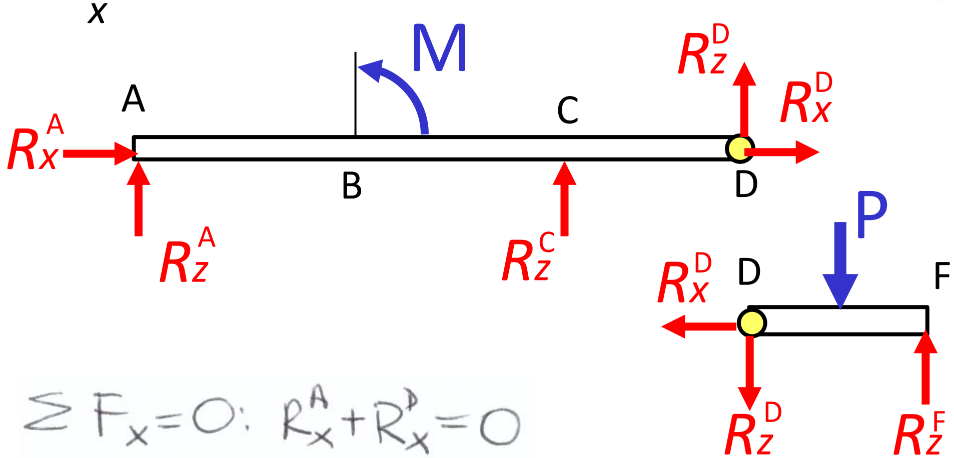
Zadanie 6



$M=1\ kNm$

$P=1\ kN$

$a=1\ m$



$$\Sigma F_x=0:\quad R_x^A+R_x^D=0$$

$$R_x^A=0$$

$$\Sigma M_A=0:\quad M+R_z^C\cdot 2a+R_z^D\cdot 3a=0$$

$$R_z^C=-\frac{M}{2a}+\frac{3}{4}P$$

$$\Sigma F_z=0:\quad R_z^A+R_z^C+R_z^D=0 \rightarrow R_z^A=\frac{M}{2a}-\frac{3}{4}P+\frac{P}{2} \rightarrow R_z^A=\frac{M}{2a}-\frac{P}{4}$$

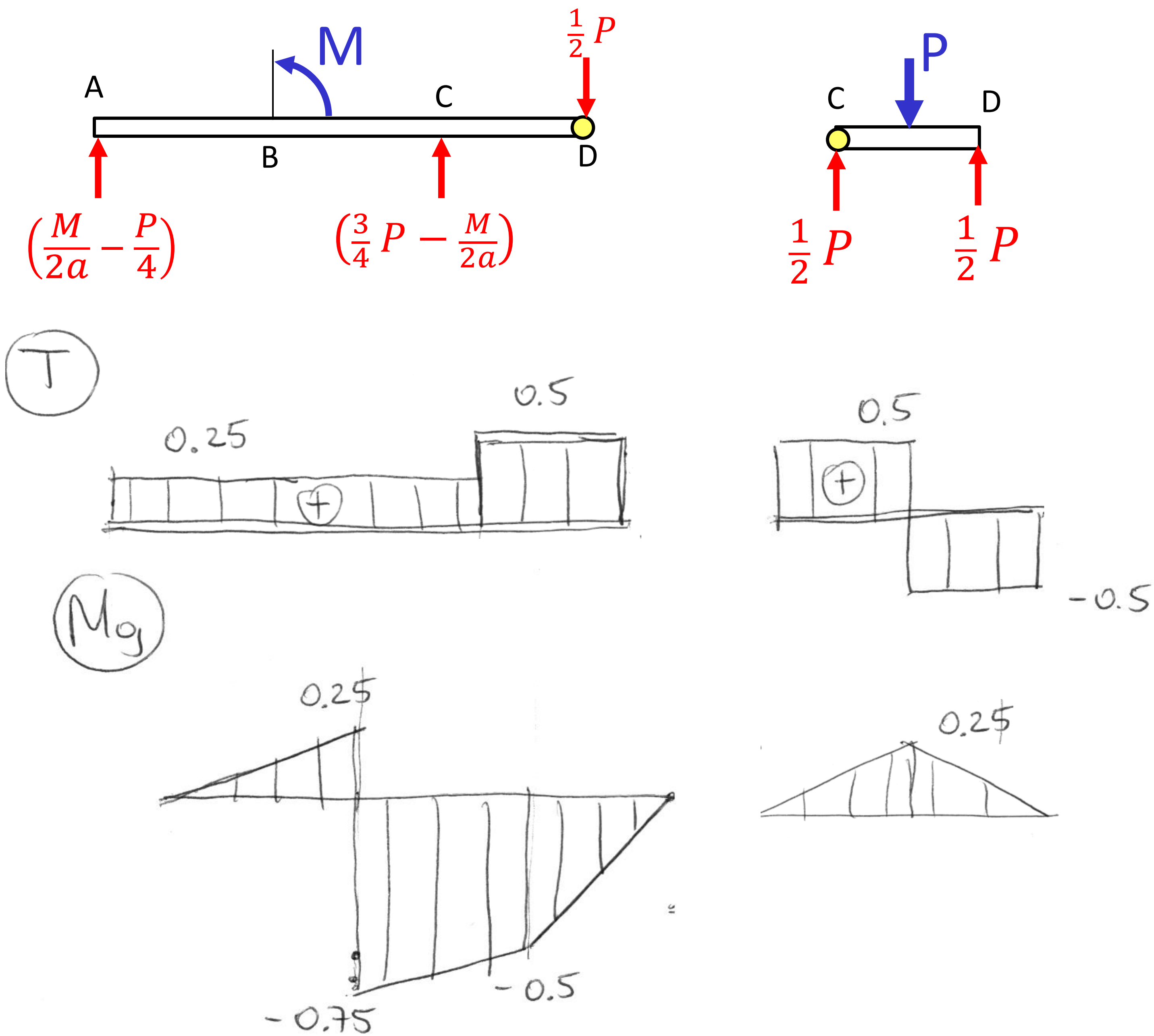
$$\Sigma F_x=0:\quad R_x^D=0$$

$$\Sigma M_D=0:\quad -P\cdot\frac{a}{2}+R_z^F\cdot a=0$$

$$R_z^F=\frac{P}{2}$$

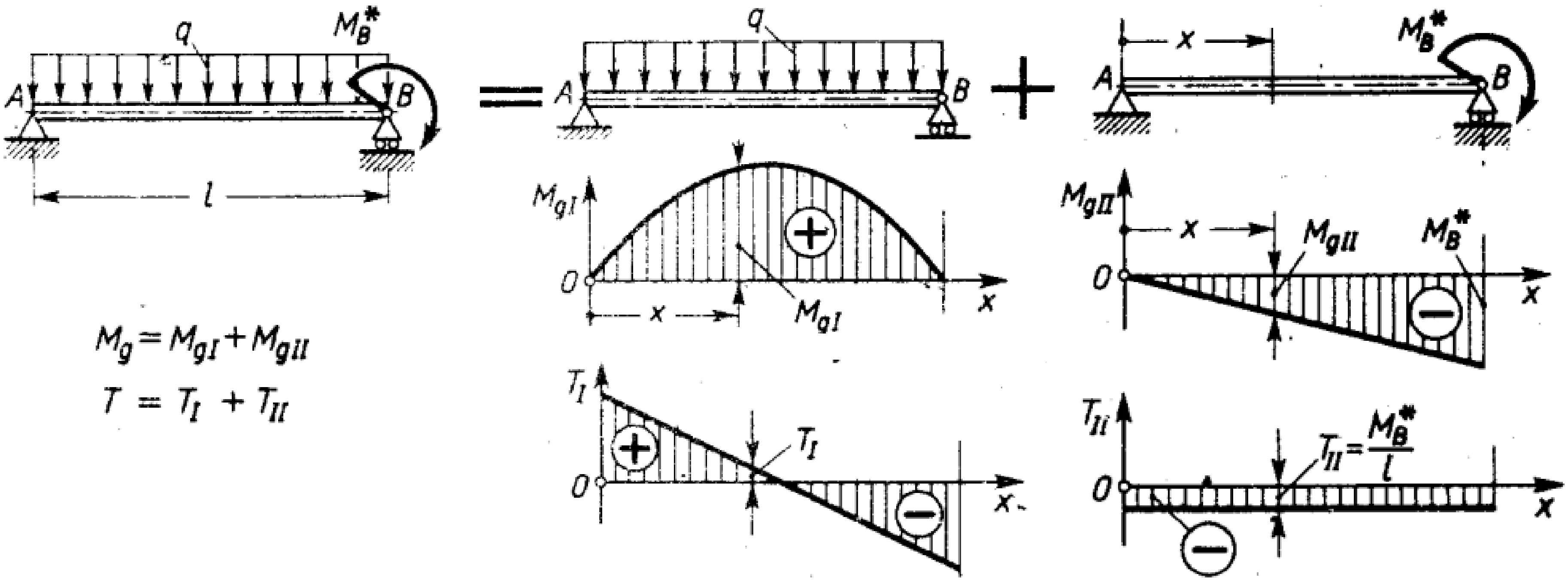
$$\Sigma F_z=0:\quad R_z^D=-P+R_z^F$$

$$R_z^D=-\frac{P}{2}$$

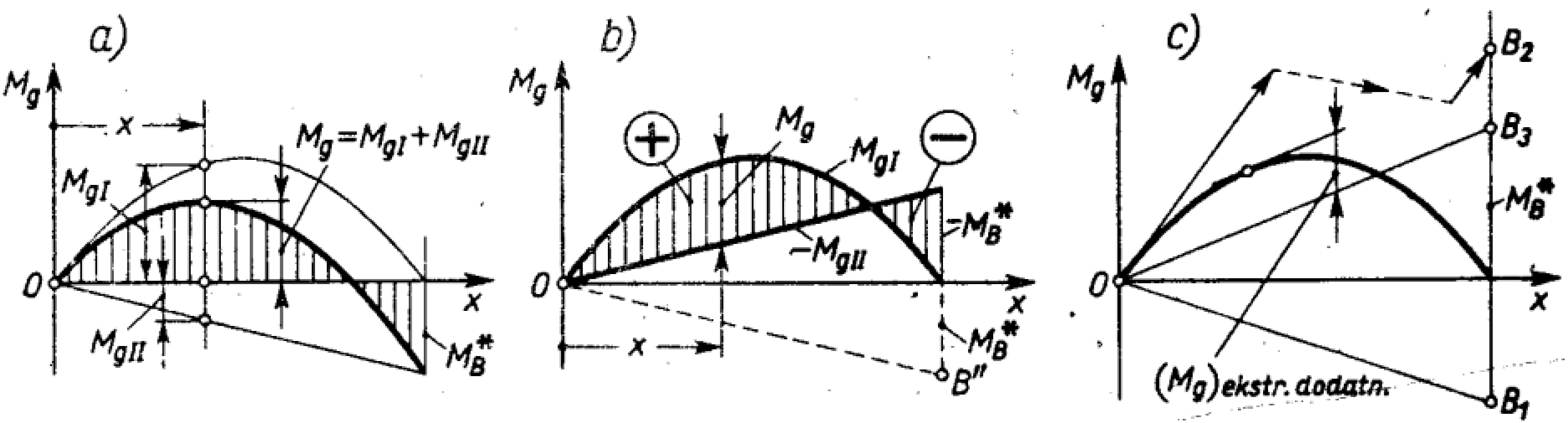


A
B
C
D
M
$\frac{1}{2}P$
$\left(\frac{M}{2a}-\frac{P}{4}\right)$
$\left(\frac{3}{4}P-\frac{M}{2a}\right)$
C
D
P
$\frac{1}{2}P$
$\frac{1}{2}P$
T
0.25
0.5
+
0.5
+
-0.5
Mg
0.25
-0.75
-0.5
0.25

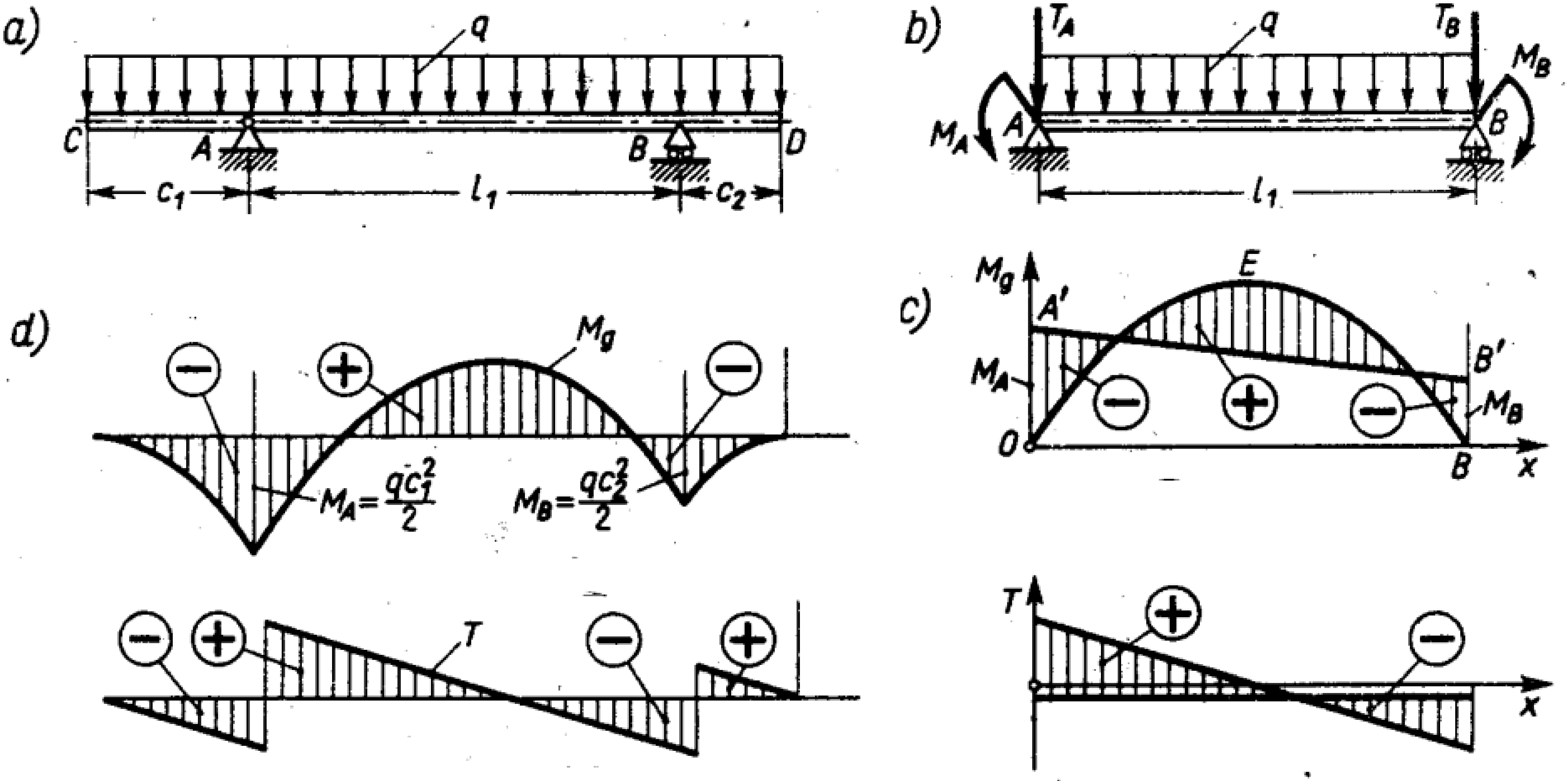
Niektóre ułatwienia obliczeń



Zastosowanie zasady superpozycji do określania M_g i T

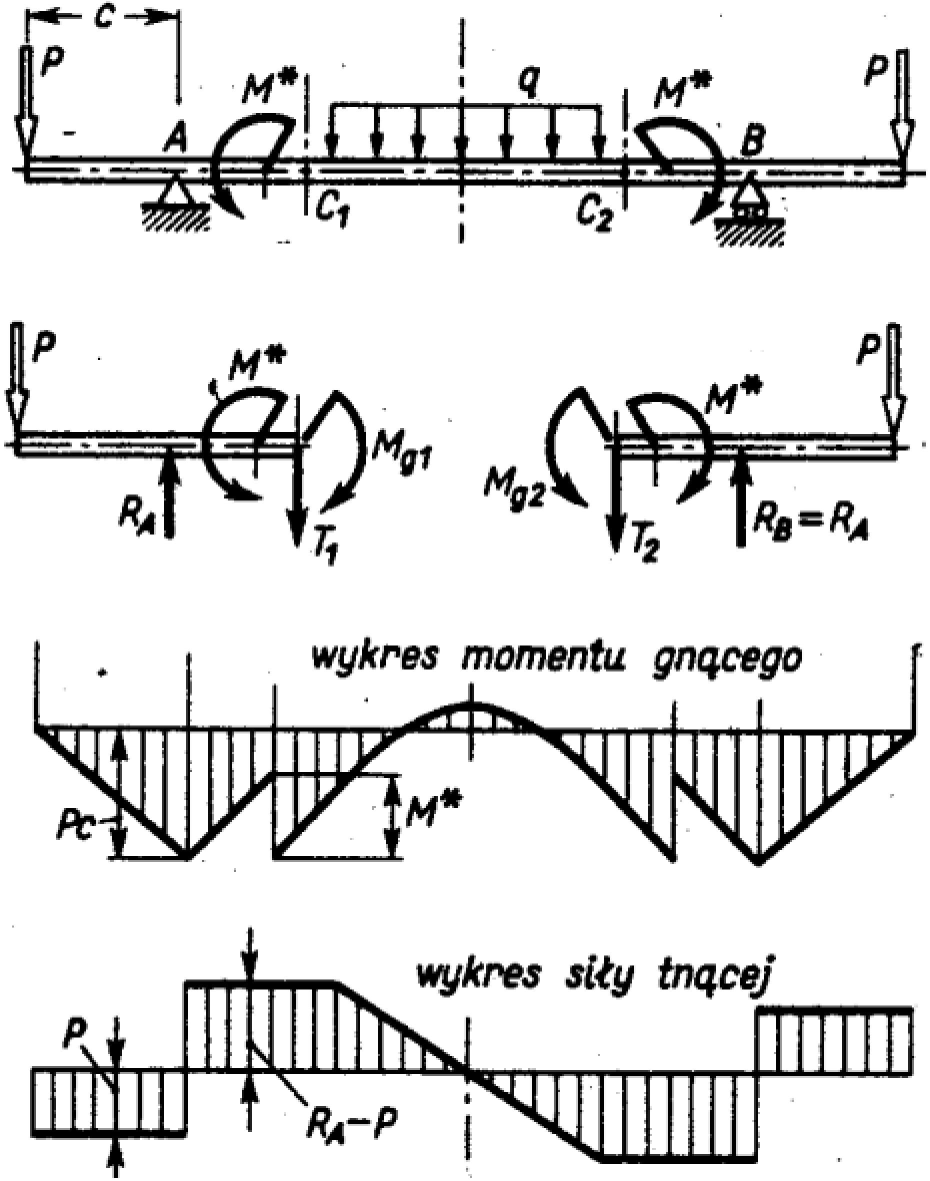


Niektóre ułatwienia obliczeń

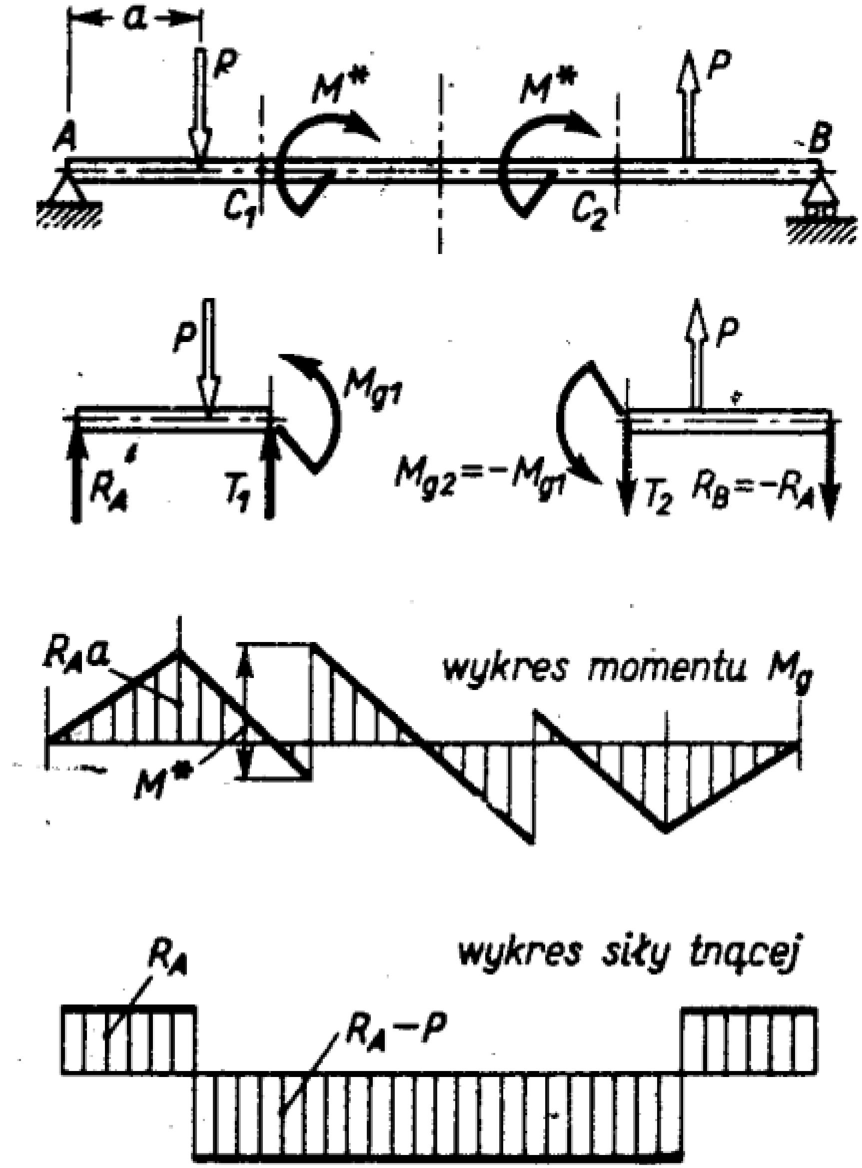


Przykład zastosowania zasady superpozycji

Niektóre ułatwienia obliczeń



Wykresy T i M_g przy obciążeniu symetrycznym



Wykresy T i M_g przy obciążeniu antysymetrycznym